Number Theory
M.Math.IInd year
Midsemestral exam 2016
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BE BRIEF!

Q 1.
(a) Show that the equation $a(a+6) = 317b - 7$ has no solutions in integers a, b.
(b) If $p > 5$ is a prime, prove that $(p-2)! - 1$ is a multiple of p but is not a power of p.

Q 2. For a positive integer $n > 2$, show that the number $\Psi(n)$ of positive integers $a \leq n$ such that $(a(a+1), n) = 1$ equals $n \prod_{p|n}(1 - 2/p)$ where the product is over the primes dividing n.

Q 3.
(a) Let p be a prime $\equiv 1$ mod 4 such that $2p+1$ is also prime. Show that 2 is a primitive root modulo $2p+1$.
(b) Let p be a prime $\equiv 3$ mod 4 such that $2p+1$ is also prime. Prove that $2p+1$ divides $2^p - 1$.

Q 4.
(a) Use the quadratic reciprocity law to characterize all primes p such that -5 is a square mod p.
Hint: Get the answer in terms of residue classes mod 20.
(b) Use the quadratic reciprocity law to prove for any prime $p > 3$ that

$$\left(\frac{3}{p}\right) = \prod_{r=1}^{[(p-1)/2]} (3 - 4\sin^2(2\pi r/p)).$$

Q 5. Let $p = a^2 + b^2$ be a prime $\equiv 1$ mod 4. Prove that $\left(\frac{a+b}{p}\right) = (-1)^{((a+b)^2-1)/8}$.

Q 6.
(a) If $a^3 + b^3 = c^3$ for integers a, b, c, show that $3|abc$.
Hint: If $(3, abc) = 1$, look at the expressions mod 9.
(b) If p is an odd prime such that $a^p + b^p \equiv c^p$ mod p^2, and $(p.abc) = 1$, prove that there exists an integer d such that $a^p + b^p \equiv d^p$ mod p^3.